IVO'S NOTES – 4/24/07 ①

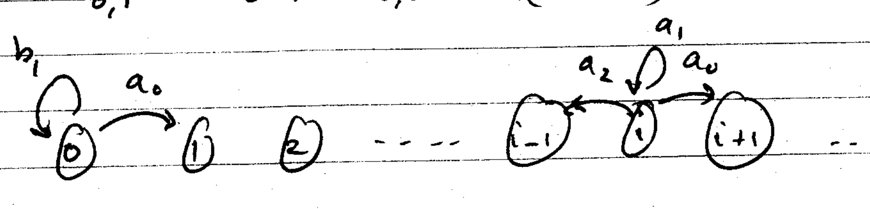
Discrete time M/M/1. or birth-and-death MC.

MC with states $\{0,1,2,\dots\}$ and

$P_{i,i+1} = a_0$, $P_{i,i-1} = a_2$, $P_{i,i} = 1 - a_0 - a_2 =: a_1$

$P_{0,1} = a_0$, $P_{0,0} = b_1 (= 1 - a_0)$

$\boxed{\text{Assume } a_0 < a_2}$



$$P = \begin{bmatrix} b_1 & a_0 & & 0 \\ a_2 & a_1 & a_0 & \\ & a_2 & a_1 & a_0 \\ 0 & & & \ddots \end{bmatrix}$$

Example M/M/1 with arrival rate λ ~~and~~ service rate μ. Take $\Delta > 0$ small. Then consider # jobs in system at $t = 0, \Delta, 2\Delta, \dots$ In interval Δ there is an arrival with prob $\Delta\lambda =: a_0$ and a departure with prob. $\Delta\mu =: a_2$

Let p_i denote steady state prob. What is p_i?

Note: $\frac{p_j}{p_i}$ = expected # visits to j before first return to i, given that P starts in i.

Define $r_i^{(k)}$ = expected # visits to $i+k$ before first return to i, given that P starts in i.

Because transition prob. are homogeneous and P is skip-free to the right, we have that

$r_i^{(k)}$ does not depend on i! So: $r_i^{(k)} = r^{(k)}$; Let $r := r^{(1)}$

Further: $p_{i+1} = p_i r^{(1)} = p_i r = \dots = p_0 r^{i+1}$, $i = 0, 1, \dots$

What is r? I.e. how to determine r?

Note: since $p_i \to 0$, it follows $r < 1$.

Since each excursion reaching $i+k$ first has to pass $i+k-1$ (skip-free!), we have:

$$r^{(k)} = r^{(k-1)} \cdot r^{(1)} = r^{(k-1)} \cdot r = \dots = r^k$$

Condition on the last state visited before visiting $i+1$, we get:

$$r^{(1)} = 1 \cdot a_0 + r^{(1)} \cdot a_1 + r^{(2)} a_2$$

So, since $r^{(1)} = r$,

$$\boxed{r = a_0 + r a_1 + r^2 a_2} \quad (1).$$

r is characterized as <u>smallest non-negative sol.</u> of (1):

<u>Proof</u>: Let x be smallest non-negative sol. of (1).
How to find x? Consider iteration

$$x(n+1) = a_0 + x(n) a_1 + x^2(n) a_2, \qquad x(0) = 0.$$

Then $x(n) \uparrow$ and $x(n) \le r$ (by induction).
So $\lim_{n \to \infty} x(n) =: x(\infty)$ exists, and
$x(\infty)$ is less or equal to any nonnegative sol. of (1).
Hence: $x = x(\infty)$. and $x \le r$.
But also $x \ge r$:
Define $r^{(k)}(n)$ = expected # visits to $i+k$ before first return to i over n periods, given that P starts in i.
Thus $r^{(k)}(n) \uparrow r^{(k)}$ as $n \to \infty$.
Also:

$$r^{(k)}(n) = \sum_{m=0}^{n} \underbrace{P^{(m)}_{i,i+k-1}}_{\text{m-steps transition prob. of } P} r^{(1)}(n-m) \le \sum_{m=0}^{n} P^{(m)}_{i,i+k-1} \cdot r^{(1)}(n) = r^{(k-1)}(n) \cdot r^{(1)}$$

Hence: $r^{(k)}(n) \le (r^{(1)})^k$.

Now $r^{(1)}(0) = 0 \le x(0)$

Suppose $r^{(1)}(n) \le x(n)$.

Then $r^{(1)}(n+1) = a_0 + r^{(1)}(n)\, a_1 + r^{(2)}(n)\, a_2$

$\le a_0 + r^{(1)}(n)\, a_1 + (r^{(1)}(n))^2 a_2$

$\le a_0 + x(n)\, a_1 + (x(n))^2 a_2$

$= x(n+1)$

Since $r^{(1)}(n) \uparrow r^{(1)} =: r$, we have $r \le x$ and thus $\underline{r = x}$ □

This also gives a mean to determine r:

Let $r(0) = 0$, $r(n+1) = a_0 + r(n)\, a_1 + r^2(n)\, a_2$, $n = 0, 1, \ldots$

Then $r(n) \uparrow r$ as $n \to \infty$.

<u>Remark</u>: we only used <u>homogeneous</u> transition prob. and <u>skip-free to the right</u>.

Hence it also works for

$$P = \begin{pmatrix} b_1 & a_0 & & \\ b_2 & a_1 & a_0 & 0 \\ b_3 & a_2 & a_1 & a_0 \\ \vdots & & & \ddots \end{pmatrix}, \quad \sum_{i=0}^{\infty} a_i = 1, \quad b_j := 1 - \sum_{i=0}^{j-1} a_i$$

"G/M/1" MC's (i.e. embedded MC of the G/M/1)

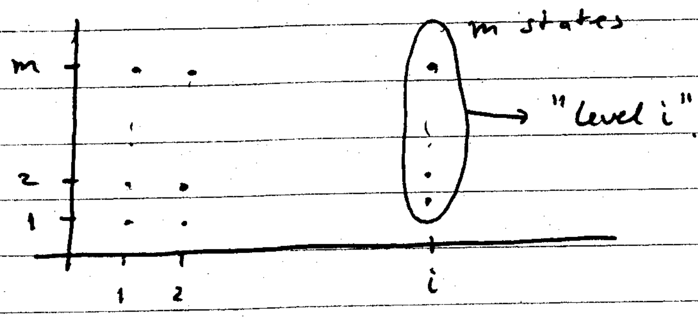
Then r is ~~the~~ minimal non-negative solution of

$$r = a_0 + r a_1 + r^2 a_2 + \cdots = \sum_{i=0}^{\infty} r^i a_i$$

Now replace every state by <u>a set</u> of states (of same size)

<u>Quasi</u> birth-and-death process (QBD) or M/M/1-type model

state space:



Irreducibel MC with states

$$\{\underbrace{(0,1), \ldots, (0,m)}_{\text{level } 0}, \underbrace{\overset{\text{phase } 1}{(1,1)}, \ldots, \overset{\text{phase } m}{(1,m)}}_{\text{level } 1}, \ldots \}$$

and transition prob. matrix

$$P = \begin{bmatrix} B_{00} & A_0 & & & & \\ B_{10} & A_1 & A_0 & & 0 & \\ 0 & A_2 & A_1 & A_0 & & \\ 0 & 0 & A_2 & A_1 & A_0 & \\ & & & \ddots & & \end{bmatrix}$$

repeating block structure

All blocks of size $m \times m$.

Note that $A_2 + A_1 + A_0$ is also MC (i.e. transition prob. matri
This MC describe behavior within level (between phases)

Assume P is irreducible and $A_2 + A_1 + A_0$ has 1 communicatin class.

When is P stable?

Stability (Neut's mean drift condition)

P is positive recurrent if and only if

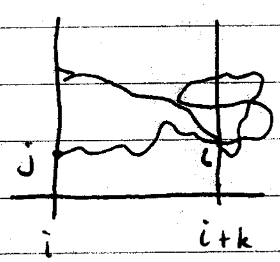
$$\underbrace{\pi A_0 e}_{\text{drift to right}} < \underbrace{\pi A_2 e}_{\text{drift to left}} \quad \text{or:} \quad \underbrace{1 \cdot \pi A_0 e + 0 \cdot \pi A_1 e - 1\, \pi A_2 e}_{\text{mean step size}} < 0 \quad (e = (1$$

where π is equilibrium distr. of $A_2 + A_1 + A_0$: $\pi(A_2 + A_1 + A_0) =$
$\pi e = 1$.

Assume P is stable

Let $p(i,j)$ denote steady-state prob. of state (i,j).

Define: $R_i^{(k)}(j,l)$ = expected # visits to $(i+k, l)$ before first return to level i, given that P starts in (i,j).



Since P is homogeneous and skip-free to the right: $R_i^{(k)} = R^{(k)}$ does not depe on i!

and
$$p(i+1, l) = \sum_{j=1}^{m} p(i,j) \cdot R^{(1)}(j,l)$$

or: $$\underbrace{p_{i+1}}_{\text{vector}} = (p(i+1,1), \cdots, p(i+1,m)) = p_i R^{(1)} = p_i R = \cdots = p_0 R^{i}$$
$$R := R^{(1)}$$

Note: since $p_i \to 0$, $i \to \infty$ it follows $\sigma(R) < 1$ — spectral radius.

How to determine R?

We have $R^{(k)} = R^{(k-1)} \cdot R^{(1)} = \dots = (R^{(1)})^k = R^k$.

and (condition on last state visited before visiting $(i+1, \ell)$)

$$R^{(1)} = A_0 + R^{(1)} A_1 + R^{(2)} A_2$$

so $$\boxed{R = A_0 + R A_1 + R^2 A_2} \quad (2).$$

Just as before (same proof!): R is minimal non-negative solution of (2).

and R can be determined by iteration:

$$R(0) = 0, \quad R(n+1) = A_0 + R(n) A_1 + (R(n))^2 . A_2, \quad n = 0, 1, 2, \dots$$

Then $R(n) \uparrow R$ as $n \to \infty$. (3)

<u>Note</u>: alternative: $R(I - A_1) = A_0 + R^2 A_2$

so $$R = (A_0 + R^2 A_2) \underbrace{(I - A_1)^{-1}}_{\text{exists } = I + A_1 + A_1^2 + \dots}$$

So: $R(0) = 0, \quad R(n+1) = (A_0 + R^2(n) A_2)(I - A_1)^{-1}, \quad n = 0, 1, \dots$

Usually converges faster.

<u>Finally</u>: P_0 follows from boundary eqs.

$$P_0 B_{00} + P_1 B_{10} = P_0$$

$\xrightarrow{P_1 = P_0 R}$ $$P_0 \underbrace{[B_{00} + R B_{10}]}_{\text{equilibrium eqs. of MC embedded on level 0}} = P_0$$

$RB_{10}(j, \ell)$ = prob. that P returns to $(0, \ell)$ from excursion starting in $(0, j)$

<u>Note</u>: It also works for

more complex boundary behavior.

$$P = \begin{bmatrix} B_{00} & A_0 & & 0 \\ B_{10} & A_1 & A_0 & \\ B_{20} & A_2 & A_1 & A_0 \\ \vdots & & & \ddots \end{bmatrix} \quad \text{or:} \quad P = \begin{bmatrix} \boxed{\begin{matrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{matrix}} & A_0 & & 0 \\ B_{20} & A_2 & A_1 & A_0 \\ \vdots & & & \ddots \end{bmatrix}$$

B_{00} : $n \times n$
B_{01} : $n \times m$
B_{i0} : $m \times n$
A_i : $m \times m$

* Then: R is minimal non-negative solution of

$$R = \sum_{i=0}^{\infty} R^i A_i$$

* Drift condition: $\pi A_0 e < \pi (A_1 + 2A_2 + \dots) e = \pi \sum_{i=1}^{\infty} i A_i e$

or: $\pi \underbrace{\sum_{i=0}^{\infty} (1-i) A_i e}_{\text{mean step size.}} < 0.$

Now: MC in <u>continuous</u> time (MP)

$$\text{generator } Q = \begin{bmatrix} B_{00} & A_0 & & 0 \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ 0 & & & \ddots \end{bmatrix}$$

(note that $A_2 + A_1 + A_0$ is also generator)

To find $p = (p_0, p_1, \cdots)$ consider

$$P := I + \Delta Q, \quad \text{where } \Delta \geq 0, \text{ sufficiently small such that } P \geq 0.$$

Then P is stochastic matrix and P and Q have the <u>same</u> equilibrium distr:

$$p = pP \Leftrightarrow p = p(I + \Delta Q) \Leftrightarrow 0 = \Delta p Q \Leftrightarrow 0 = pQ$$

This also implies that P is positive recurrent iff Q is positive recu

$$P = \begin{bmatrix} I + \Delta B_{00} & \Delta A_0 & & 0 \\ \Delta A_2 & I + \Delta A_1 & \Delta A_0 & \\ & \Delta A_2 & I + \Delta A_1 & \Delta A_0 \\ 0 & & & \ddots \end{bmatrix}$$

<u>Drift condition</u>: $\pi \Delta A_0 e < \pi \Delta A_2 e$, where $\pi(I + \Delta(A_2 + A_1 + A_0)) = \pi$

or: $\pi(A_2 + A_1 + A_0) = 0$

$\pi e = 1$

<u>p_i</u>: $p_i = p_0 R^i$ where R is min. non-neg. sol. of:

$$R = \Delta A_0 + R(I + \Delta A_1) + R^2 \Delta A_2$$

so

$$0 = A_0 + R A_1 + R^2 A_2.$$

Determine R via (3) or via: $R = (A_0 + R^2 A_2)\underbrace{(-A_1)^{-1}}_{\geq 0\,!}$

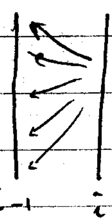
Special cases of M/M/1-type (or QBD) models

(1) $$A_2 = V.\alpha = \begin{pmatrix} V_1 \alpha \\ V_2 \alpha \\ \vdots \\ V_m \alpha \end{pmatrix}, \quad \alpha = (\alpha_1, \ldots, \alpha_m), \quad \alpha e = 1$$

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix}$$

so rows of A_2 are the same upto scaling.

This means: given that P jumps to a lower level, the entrance distribution is always the same, i.e. it does not depend on which state P came from.



What is entrance distr.? α!

Note that: V_j is prob. that P jumps to lower level from (i,j)

So $$R = A_0 + RA_1 + \underbrace{R^2 A_2}_{RA_0 e . \alpha} \rightarrow R\left[I - A_1 - A_0 e\alpha\right] = A_0$$

$$R = A_0\left[I - (A_1 + A_0 e\alpha)\right]^{-1}$$

(in continuous time: $R = -A_0(A_1 + A_0 e\alpha)^{-1}$)

Alternative derivation:

$$P_i = P_{i-1}A_0 + P_i A_1 + P_{i+1}A_2 \overset{A_2 = V\alpha}{=} P_{i-1}A_0 + P_i A_1 + P_{i+1} V \alpha$$

Balance flow between level i and level i+1:

$$P_i A_0 e = P_{i+1} A_2 e = P_{i+1} V$$

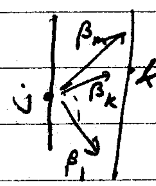
Hence
$$P_i = P_{i-1}A_0 + P_i A_1 + P_i A_0 e \alpha$$

so
$$P_i = P_{i-1} \underbrace{A_0\left[I - A_1 - A_0 e \alpha\right]^{-1}}_{R.}$$

(2) $A_0 = w\cdot\beta = \begin{pmatrix} w_1\\ \vdots \\ w_m\end{pmatrix}(\beta_1,\dots,\beta_m)$, $\quad \beta e = 1$

This means: if P jumps to higher level, then the entrance distr. is always the same, i.e., it does not depend on which state P came from.
(note that w_j is the prob. that P jumps to higher level from (i,j))

This implies:



$$R(j,l) = (1-w_j)\cdot 0 + w_j \cdot \underbrace{\sum_{k=1}^{m} \beta_k S(k,l)}_{\substack{\text{only depends on } l \\ =:\, a_l}}$$

where $S(k,l)$ = expected # visits to $(i+1,l)$ before first return to level i, given that P starts in $(i+1,k)$.

So R is of the form:

$$R = w\cdot a \quad \text{for some vector } a=(a_1,\dots,a_m).$$

Alternative derivation: look at iteration scheme

$$R(n+1) = A_0 + R(n)A_1 + R^2(n)A_2, \qquad A_2 = w\cdot\beta$$

Then $R(1) = A_0$, so it is of the form $R(1) = w\cdot a_1$
(so $R^2(1) = (a_1 w)\cdot w a_1$)

and $R(2) = w\big[\underbrace{\beta + a_1 A_1 + (a_1 w) a_1 A_2}_{=:\, a_2}\big] = w a_2$

$\vdots$

$R(n) = w a_n \uparrow R$, so also R is of the form $R = w\cdot a$.

Thus $R^i = (aw)^{i-1} R = \eta^{i-1} R$ where $\eta := aw$

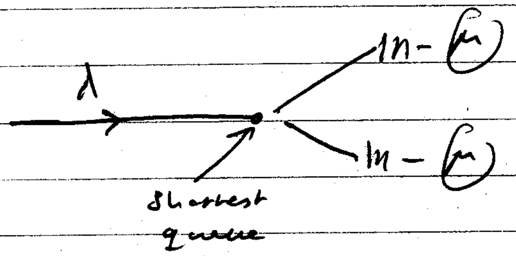
so $p_i = p_0 R^i = \eta^{i-1} p_0 R = \eta^{i-1} p_1$, $i = 1, 2, \dots$

and η is unique root on $(0,1)$ of: $\det\left(A_0 + (A_1 - I)\eta + A_2\eta^2\right) = 0$

or determine a via: $a_{n+1} = \beta + a_n A_1 + (a_n w) a_n A_2$, $n = 0, 1, \dots$, $a_0 = 0$

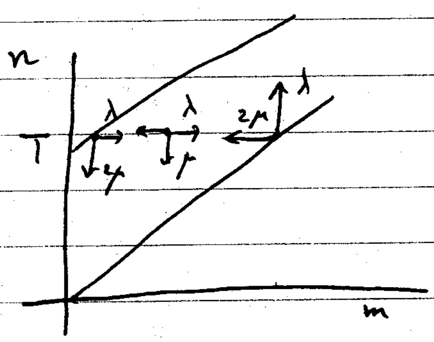
Examples

(1) Shortest queue with threshold jockeying



If difference between longest and shortest queue $\geq T$, then job jumps from longest to shortest queue.

States (m, n) where m = length of shortest queue
n = length of longest queue
(so $m \leq n$).



Levels? Take level $n = \{ (n-T, n), (n-T+1, n), \ldots, (n,n)$

and put states (m,n) with $n < T$ into set of boundary states

Then $A_0 = \begin{pmatrix} & 0 & \\ 0 \ldots & 0 \ \lambda & 0 \end{pmatrix}$: only 1 nonzero row!

Hence $P_n = (P(n-T, n), \ldots, P(n,n)) = \eta^{n-T} \cdot P_T$, $n = T, T+1, \ldots$

What is η? Let $V_\ell = \{ (m,n) \mid m+n = \ell \}$

Then balancing flow between V_ℓ and $V_{\ell+1}$ gives

$$P(V_{\ell+1}) \cdot 2\mu = P(V_\ell) \cdot \lambda \qquad (\ell \geq T).$$

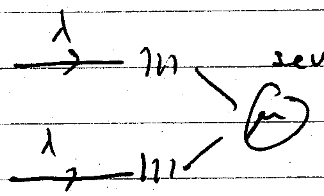
So $P(V_{i+2}) = \rho^2 P(V_i)$

Also $p(m+1, n+1) = \eta \cdot p(m,n)$ and thus

$$P(V_{i+2}) = \eta\, P(V_i)$$

Conclusion : $\eta = \rho^2$.

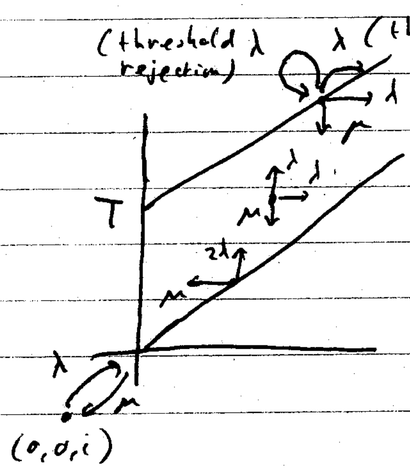
(2) Longest queue with threshold addition/rejection



If diff between longest queue and shortest queue $> T$, then:

(i) add job to shortest queue

or: (ii) reject job in longest queue



States (m,n),

m = length shortest queue

n = length longest queue

$(0,0,i)$ = queues are empty, server idle.

Levels? Take level $m = \{(m,m), (m,m+1), \ldots, (m, m+T)\}$,

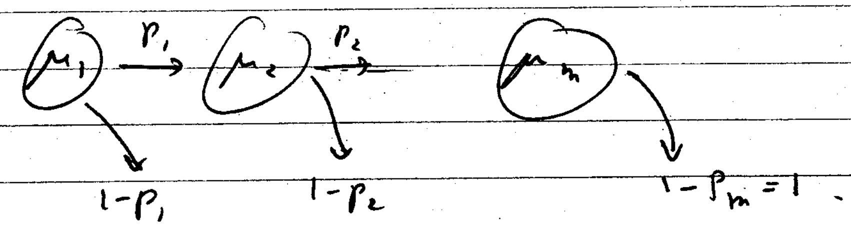
Then
$$A_2 = \begin{pmatrix} 0 & \mu & 0 & & 0 \\ & & 0 & & \end{pmatrix} \text{ ; only 1 non-zero row!}$$

$= v \cdot \alpha$ where $v = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\alpha = (0, 1, 0, \ldots$

Hence
$$R = -A_0 (A_1 + A_0 e \alpha)^{-1}.$$

(3) $M/C_m/1$ queue

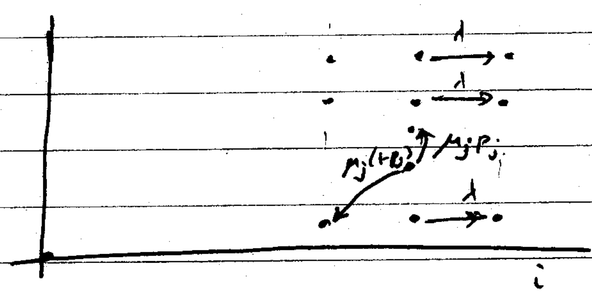
C_m :



Class of distributions C_m is <u>dense</u>:

every (continuous) distr. on $(0,\infty)$ can be approximated arbitrarily close by C_m-distr.

States (i,j) : i = # jobs in system
j = phase of job in service, $j = 1, \dots, m$
0 : idle (empty) state



level $i = \{(i,1), \dots, (i,m)\}$, $i \geq$

level $0 = \{(0,0)\}$

So: $A_0 = \lambda I$, $A_1 = \begin{pmatrix} -(\lambda+\mu_1) & \mu_1 p_1 & & 0 \\ & \ddots & \ddots & \\ & & & \mu_{m-1} p_{m-1} \\ 0 & & & -(\lambda+\mu_m) \end{pmatrix}$

$$A_2 = \begin{pmatrix} \mu_1(1-p_1) & & \\ \vdots & & 0 \\ \mu_m(1-p_m) & & \end{pmatrix} = v \cdot \alpha \quad \text{where } v = \begin{pmatrix} \mu_1(1-p_1) \\ \vdots \\ \mu_m(1-p_m) \end{pmatrix}$$

$\alpha = (1, 0, \dots, 0)$

Hence: $R = -A_0(A_1 + A_0 e \alpha)^{-1}$!